Lecture 8 - Gauss's Law

A Puzzle...

Example

Consider an infinite number of identical point charges q are placed on the x-axis at $x = 1$, $x = 2$, $x = 3$... What is the electric field at the origin?

Solution

Adding the effects of each point charge yields

$$\vec{E} = (-\hat{x})\, k\, q \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) \tag{1}$$

one of the most famous mathematical series. The term in parentheses equals $\frac{\pi^2}{6}$. An incredibly elegant geometric proof of that relation can be found using this unbelievably clever method to compute this sum! □

Example

Calculate the potential energy, per ion, for an infinite 1D ionic crystal with separation a; that is, a row of equally spaced charges of magnitude e and alternating sign.
Hint: The power-series expansion of Log[$1 + x$] may be of use.

Solution

Suppose the array is built inwards from the left (that is, from negative infinity) as far as a particular ion. To add the next positive ion on the right, the amount of external work required equals

$$-\frac{k e^2}{a} + \frac{k e^2}{2 a} - \frac{k e^2}{3 a} + \cdots = -\frac{k e^2}{a}\left(1 - \frac{1}{2} + \frac{1}{3} + \cdots\right) \tag{2}$$

The terms in parenthesis look remarkably similar to the Taylor series of Log[$1 + x$] when $x = 1$. (What is the Taylor series of Log[$1 + x$]? $\frac{d}{dx}$Log[$1 + x$] $= \frac{1}{1+x} = 1 - x + x^2 - \cdots$, and integrating both sides yields Log[$1 + x$] $= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$, since the right hand side is a polynomial expansion, it is also the Taylor series of Log[$1 + x$]. This Taylor series is converging for $-1 < x \leq 1$ (with convergence on $-1 < x < 1$ assured by the alternating series test)). Therefore the work required to bring this charge in equals $-\frac{k e^2}{a}$ Log[2], which equals the energy of the infinite chain *per ion*.

The addition of further particles on the right doesn't affect the energy involved in assembling the previous ones, so this result is indeed the energy per ion in the entire infinite (in both directions) chain. The result is negative, which means that it requires energy to move the ions away from each other. This makes sense, because the two nearest neighbors are of opposite sign.

Note that this is an exact result! It does not assume that a is small. Getting such a nice closed form is much more difficult in 2 and 3 dimensions. □

Theory

Visualizing the Electric Field (Extended)

Electric Field Lines

Advanced Section: Escaping Field Lines

Math Background: What is a Surface Integral?

We are already familiar with surface integrals of scalar functions. For example, the surface area of a sphere with radius R centered at the origin is given by

$$\int_{\text{surface}} da = \int_0^{2\pi} \int_0^{\pi} R^2 \,\text{Sin}[\theta] \, d\theta \, d\phi = 4\pi R^2 \tag{5}$$

Although we use spherical coordinates in this particular problem, we could also have used Cartesian coordinates (with $da = dx\,dy$) or any other coordinate system.

More generally, suppose we are given a function $f[\theta, \phi]$ defined everywhere on the surface of the spherical shell of radius R. What is the integral of $f[\theta, \phi]$ across the entire spherical shell? This requires a very minor modification to the formula above, namely,

$$\int_{\text{surface}} f[\theta, \phi] \, da = \int_0^{2\pi} \int_0^{\pi} f[\theta, \phi] \, R^2 \,\text{Sin}[\theta] \, d\theta \, d\phi \tag{6}$$

We would need to know the specific function $f[\theta, \phi]$ to explicitly evaluate this integral, but we observe that the surface area is merely a specific case of Equation (6) with $f[\theta, \phi] = 1$ everywhere on the sphere.

Lastly, rather than being given a scalar function $f[\theta, \phi]$, we could be given a vector function $\vec{F}[\theta, \phi]$ defined everywhere on the sphere. In such a case, we define the surface integral to be

$$\int_{\text{surface}} \vec{F}[\theta, \phi] \cdot d\vec{a} = \int_{\text{surface}} \left(\vec{F}[\theta, \phi] \cdot d\hat{a}\right) da \tag{7}$$

No need to panic! The right hand side is simply the same surface integral from Equation (6), except that now the scalar function is the dot product of two vectors. The first vector $\vec{F}[\theta, \phi]$ is the function that you are integrating over the sphere. The second vector $d\hat{a}$ is a unit vector of an infinitesimal patch in the direction normal to the surface. On the sphere, the unit normal vector at any point is given by $d\hat{a} = \hat{r}$, so that we can write the surface integral as

$$\int_{\text{surface}} \left(\vec{F}[\theta, \phi] \cdot d\hat{a}\right) da = \int_0^{2\pi} \int_0^{\pi} \left(\vec{F}[\theta, \phi] \cdot \hat{r}\right) R^2 \,\text{Sin}[\theta] \, d\theta \, d\phi \tag{8}$$

For closed surfaces, we take the unit normal to be pointing outwards by convention. For open surfaces, you can arbitrarily choose between the two possible directions that the unit vector can point (if you switch, it will flip the sign of the surface integral).

Usually, we will be working in setups where the unit normal vector is obvious. For example, if we have a sheet in the x-y plane, then the unit normal vector will be $\hat{z}$ (or $-\hat{z}$, you can choose either as long as you are consistent throughout your surface integral) at all points. Similarly, if we have a cylinder whose axis lies on the z-axis, then the unit normal vector at its top cap will be $\hat{z}$ (remembering that unit normal vectors point outwards for closed surfaces) and $-\hat{z}$ at its bottom cap. Finally, in cylindrical coordinates (ρ, θ, z), the unit vector will be $\hat{\rho}$ for all points along the curved surface of the cylinder.

Gauss's Law

An incredibly useful and beautiful result, Gauss's Law is definitely worth memorizing! Here we write it for a discrete and continuous charge distribution.

The integral $\int \vec{E} \cdot d\vec{a}$ over the surface, equals $\frac{1}{\epsilon_0}$ times the total charge enclosed by the surface,

$$\int \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \sum_j q_j = \frac{1}{\epsilon_0} \int \rho \, dv \tag{9}$$

For a combination of both (for example, a point charge near an infinite sheet), the Principle of Superposition tells us that we sum over the discrete charges and integrate over the charge distributions within our surface.

Example

An infinite line of charge with uniform charge density λ lies on the z-axis. Determine:

- The direction of the electric field at any point
- The magnitude of the electric field at any point
- Double check this result using both Gauss's Law and straight-up integration

Solution

By symmetry, the electric field must point radially outwards from the line of charge.

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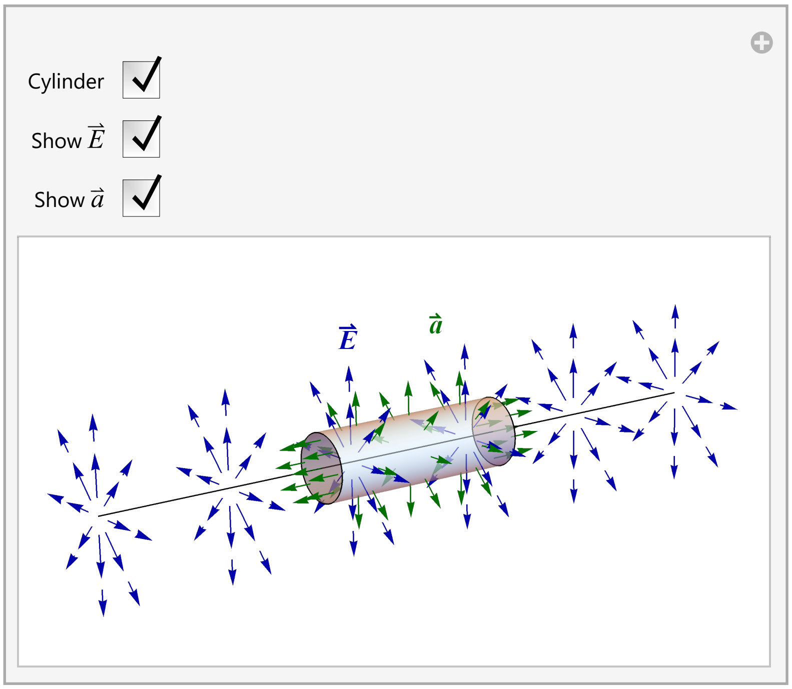

To determine the magnitude of the electric field, we use a Gaussian cylinder (the object whose surface we will integrate over) whose axis lies on the line of charge and calculate $\int \vec{E} \cdot d\vec{a}$ along this cylinder. Since $\vec{E}$ points radially, $\vec{E} \cdot d\vec{a} = 0$ along the (flat) top and bottom of the cylinder, and by radial symmetry, $\vec{E} \cdot d\vec{a} = E[r]\, da$ will be a constant value everywhere along the cone (where $E[r]$ is the magnitude of the electric field at a distance r from the wire). Therefore, Gauss's Law yields

$$\int \vec{E} \cdot d\vec{a} = E[r] \int da = E[r]\,(2\,\pi\,r\,L) = \frac{\lambda\,L}{\epsilon_0} \tag{10}$$

which yields

$$E[r] = \frac{\lambda}{2\,\pi\,r\,\epsilon_0} \tag{11}$$

Alternatively, we could have computed this value by straight-up integration, choosing the radial component of the electric field

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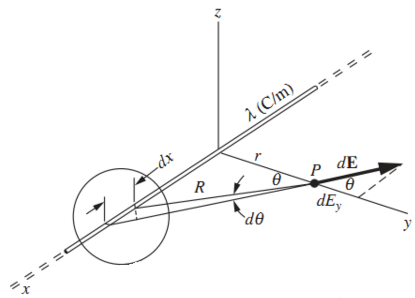

$$
\begin{aligned}
E[r] &= \int_{-\infty}^{\infty} \frac{k\,\lambda\,dx}{x^2+r^2}\,\frac{r}{\left(x^2+r^2\right)^{1/2}} \\
&= \left(\frac{k\,\lambda\,x}{r\left(x^2+r^2\right)^{1/2}}\right)_{x=-\infty}^{x=\infty} \\
&= \frac{2\,k\,\lambda}{r} \\
&= \frac{\lambda}{2\,\pi\,r\,\epsilon_0}
\end{aligned} \tag{12}
$$

This yields the same result as above, but after significantly more work. □

A Note about Symmetry

The key to the last problem was proving that for a line of charge lying on the x-axis, the electric field $\vec{E} = E[r]\,\hat{r}$ points radially away from the wire. Let's prove this explicitly.

If the infinitely long wire lies along the x-axis, then the points on the wire are given by $(x, 0, 0)$ where $x \in (-\infty, \infty)$. Let us consider the electric field at a point $(0, 0, z)$, and let us call the electric field $\vec{E} = \langle E_x, E_y, E_z \rangle$.

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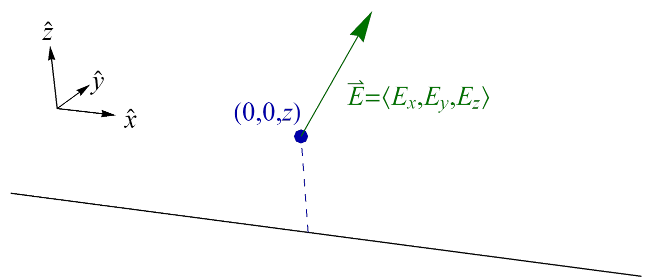

First, let us flip the setup along the $x = 0$ as shown below, which means that every point (and every vector) goes from $(x, y, z) \to (-x, y, z)$. Therefore the electric field will go from $\langle E_x, E_y, E_z \rangle \to \langle -E_x, E_y, E_z \rangle$, shown by the solid arrow → dashed arrow. However, since every point $(x, 0, 0)$ on the line of charge goes to another point $(-x, 0, 0)$ on the line of charge, and the point of interest $(0, 0, z) \to (0, 0, z)$ does not change, the physical setup of the problem remains exactly the same. Therefore, the electric field before and after the reflection must be the same, and this is only possible if $E_x = 0$.

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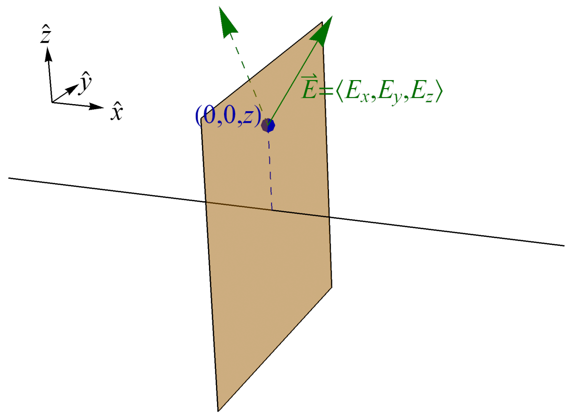

Similarly, consider a reflection about the $y = 0$ plane, so that every point goes from $(x, y, z) \to (x, -y, z)$. Therefore the electric field will go from $\langle 0, E_y, E_z\rangle \to \langle 0, -E_y, E_z\rangle$. Each point on the line of charge will not be changed $(x, 0, 0) \to (x, 0, 0)$, and similarly the point of interest will not change $(0, 0, z) \to (0, 0, z)$, so that once again the physical setup is the same and hence the electric field must be the same. This can only happen in $E_y = 0$.

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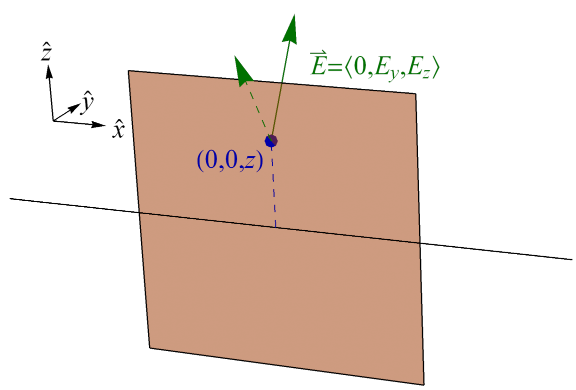

We conclude that at the point $(0, 0, z)$, the electric field point straight up (i.e. radially away) from the line of charge. We can generalize this result to find the direction of the electric field at an arbitrary point (x, y, z) using symmetry. First, we note that because the wire is infinitely long, translating along the x-direction cannot change the electric field, since we could equally well have defined our origin at some other point $(X, 0, 0)$ and the physical setup would be identical to that found above, so that the electric field at $(X, 0, z)$ must point in the $\hat{z}$ direction. Finally, we use the rotational symmetry of the setup (i.e. rotating our coordinate system about the x-axis) to find that the electric field at any point (x, y, z) points radially outward from the wire in the direction $\langle 0, y, z\rangle$.

Problems

Field from a Cylindrical Shell

Example

Consider a charge distribution in the form of an infinitely long hollow circular cylinder (like a long charged pipe) of radius R. If the cylinder has uniform charge per unit area σ, what is the electric field inside and outside the cylinder?

Solution

The electric field inside the cylinder must be pointing radially about the axis of symmetry. Therefore, using a Gaussian surface of a small cylinder with radius $r < R$ and length l placed along the axis of the large cylinder, we find that

$$E\,(2\,\pi\,r\,l) = \frac{q_{\text{in}}}{\epsilon_0} = 0 \tag{13}$$

because there is no charge inside the cylindrical shell. Thus, $E = 0$ for all $r < R$.

The electric field outside the cylinder is found using a the same Gaussian surface but with $r > R$ so that

$$E\,(2\,\pi\,r\,l) = \frac{q_{\text{in}}}{\epsilon_0} = \frac{2\,\pi\,R\,l\,\sigma}{\epsilon_0} \tag{14}$$

or equivalently $E = \frac{R}{r}\,\frac{\sigma}{\epsilon_0}$. Since the charge per unit length on the cylindrical shell equals $2\,\pi\,R\,\sigma$, $E = \frac{(2\,\pi\,R\,\sigma)}{2\,\pi\,r\,\epsilon_0}$ so that the electric field outside the cylinder is the same as if all of the charge was concentrated at on the axis of the cylinder.

These results are very interesting, but let me point out one subtlety in this result. If you ask for the electric field $E[r]$ as a function of the radial distance r from the cylindrical shell, then starting at ∞ and coming towards the axis the answer will be

$$E[r] = \begin{cases} \frac{R}{r}\,\frac{\sigma}{\epsilon_0} & r > R \\ 0 & r < R \end{cases} \tag{15}$$

This implies that at $r = R$, there is a discontinuity; the value of $E[r] \to \frac{\sigma}{\epsilon_0}$ from the outside of the cylindrical shell and $E[r] \to 0$ from the inside of the shell. This begs the question, what exactly is the electric field at $r = R$? As we will see from the electric field of an infinite plane, the answer will be $E[R] = \frac{\sigma}{2\,\epsilon_0}$, and you are encouraged to do the explicit calculation yourself! For now, we note that $E[R]$ is the *average* of the values two limits of $E[r]$ from inside and outside the cylindrical shell. □

Extra Problem: Field from a Cylindrical Shell, Right and Wrong

Mathematica Initialization